T = residence time, hr.
t = time, hr.
V = crystallizer volume, liters
β = kinetic order
ρ = particle density, g./cu. mm.
ρ_s = suspension density, g./liter
θ = dimensionless time
sub o = steady state

LITERATURE CITED

1. Bransom, S. H., W. J. Dunning, and B. Millard, *Disc. Faraday Soc.*, **5**, 83 (1949).
2. Larson, M. A., and D. C. Murray, A laboratory study of continuous mixed suspension crystallization. Preprint 14, Fiftieth Natl. Meeting, A.I.Ch.E., Buffalo, New York (1963).
3. McCabe, W. L., *Ind. Eng. Chem.*, **21**, 112 (1929).
4. Murray, D. C., Ph.D. thesis, Iowa State Univ., Ames, Iowa (1964).
5. Randolph, A. D., and M. A. Larson, Analog simulation of dynamic behavior in a mixed crystal suspension. Preprint 38, Fifty-fifth Ann. Meeting, A.I.Ch.E., Chicago, Illinois (December 2-6, 1962).
6. ———, *A.I.Ch.E. J.*, **8**, 639 (1962).
7. Robinson, J. N., and J. E. Roberts, *Can. J. Chem. Eng.*, **35**, 105 (1957).
8. Saeman, W. C., *A.I.Ch.E. J.*, **2**, 107 (1956).

Manuscript received October 7, 1964; revision received February 23, 1965; paper accepted February 26, 1965. Paper presented at A.I.Ch.E. Boston meeting.

The Numerical Solution of Boundary-Layer Problems

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A new method for numerical solution of boundary-layer problems has greater applicability as well as greater speed and accuracy than the several previously proposed methods. The method uses the Goertler-transformed equations and a nonlinear finite-difference procedure. In most earlier work on difference methods the equations have been linearized at each step parallel to the boundary. The method of treatment of the boundary conditions on the equation of change for energy or mass transfer is shown to have an important influence on accuracy, and a new, more accurate method of treating boundary conditions involving the normal derivative is presented.

Typical complete solutions are compared with the several methods. The solutions include both similar and more general flows up to separation, as well as coupling between the momentum and energy-balance equations.

The boundary-layer problems of most interest today involve high temperatures, large variations in physical properties, ionization, chemical reactions, and other phenomena that greatly complicate the application of series expansions or other classical methods of solution. Interest has increased therefore in recent years in the development of numerical methods of general applicability. One such method that holds important advantages over previously proposed methods is presented here, with a number of comparisons with results from previously proposed methods. One of the interesting results of the work is that a nonlinear finite-difference procedure was finally adopted as being much more efficient and accurate than procedures of the usual type, where the difference equations are linearized at each step in the procedure.

Hartree-Womersley Methods

A well-known class of methods is based on the procedure of Hartree and Womersley (*9*), used with some success long before the advent of the modern digital computer. This method, with certain modifications, has been used and advocated by several workers (*14, 17* to *19*). The basic idea of Hartree and Womersley was to approximate the derivatives with respect to the X direction by using divided differences and to leave the derivatives in the Y direction in differential form. Hence, the procedure involved the solution of a system of ordinary differential equations at each X step as the solution was propagated in the positive X direction. The ordinary differential equations are solved by the well-known techniques for integration of initial-value problems. The method of satisfying the boundary conditions is a great disadvantage of the Hartree-Womersley methods. Conditions at the outer edge of the boundary layer are satisfied by assuming sufficient additional conditions at the boundary to make integration possible as an initial-value problem starting at the boundary. The computed values of the functions at the outer edge are then compared with the desired values, the assumed conditions are corrected, and the procedure is repeated until the outer conditions are satisfied. This technique of satisfying the boundary conditions has been found (*17, 18*) satisfactory in classical boundary-layer problems where the momentum and energy-balance equations are uncoupled, since a single condition is imposed at the outer edge. However, since two or more exterior conditions must be satisfied simultaneously, the procedure is cumbersome in problems involving a coupled energy balance and one or more diffusion relationships.

Smith and Cutter found the Hartree-Womersley method applicable only for grid spacings where ΔX was larger than $X/25$ at all grid points; for smaller values of ΔX the solutions diverged, giving clear evidence of instability. The method cannot be considered generally applicable because of the stability problem, even though it gave reliable results in several applications. (*17, 18, 19*).

Finite Difference Methods

The second class of methods includes all those that might be designated *finite-difference methods.* All derivatives are approximated by divided differences. The solution is propagated in the positive X direction by solving the resulting system of algebraic equations at each X step. Hence, the boundary conditions are satisfied in the basic algebraic procedure. Of the many possible difference methods, prior workers on nonlinear differential equations have almost invariably selected those in which the difference equations are made linear at each step in the procedure. Such a selection, which greatly simplifies the algebraic procedure, has been found satisfactory in many cases.

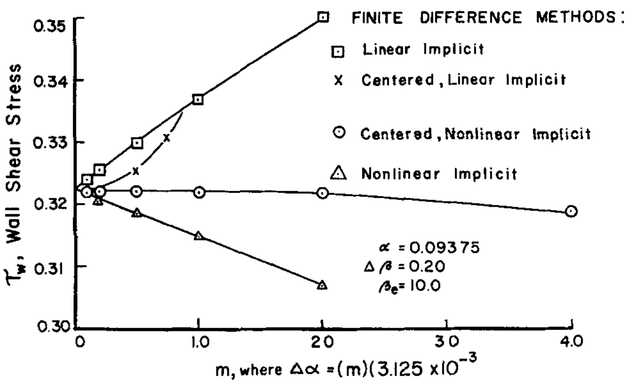

Fig. 1. Wall-shear variation with tangential step size, Goertler incompressible, steady state equations.

Explicit schemes of calculation studied and advocated by several workers (*1, 20*) appear suitable for certain problems involving a zero or a favorable pressure gradient; however, it became clear early in this work that any explicit method would be inadequate for solving problems involving an adverse pressure gradient due to the severe limitations in step size. The principal problem is, without exception, that the explicit methods involve requirements for stability of the form $\Delta X/[U\Delta Y)^2] < 1$ at all interior points. In free-convection problems, as well as in problems with adverse pressure gradients, this velocity component approaches zero at some interior points. Hence, the permissible X step size must approach zero and the schemes fail. It was therefore concluded from these preliminary studies that some implicit method would be required for general applicability.

The Goertler Transformation

Considerable attention was given to determining the form of the boundary-layer equations most suitable for numerical solution. Studies were carried out on the untransformed equations, the Crocco-transformed equations, the Howarth–Dorodnitsyn–transformed equations, and the Goertler-transformed equations. The Goertler transformation (*8*) was found to have important advantages in that the Goertler-transformed equations reduced to the Falkner-Skan or similar equations when the form of the exterior flow permitted a similarity transformation. Hence, in the Goertler plane the values of the dependent variables are independent of distance parallel to the boundary in similar flows. In more general flows the profiles and the boundary-layer thickness change relatively little with distance. This behavior greatly improves the accuracy of the numerical procedure for a given amount of computation.

In working with the untransformed equations, we found that in some cases results were strongly affected by the choice of initial profiles and by the particular method chosen for treating the continuity equation. Such a difficulty might be anticipated a priori, since, as is well known, the leading edge is a singular point in the boundary-layer equations, and, strictly speaking, there is no "correct" initial profile in the physical plane. Some workers (*1, 5*) have dealt with this difficulty by considering only flow geometrics with similar initial regions, a solution which enables them to obtain exact starting profiles at some distance removed from the singular point. Other workers (*5, 20*) have used approximations for the initial profiles at the leading edge. These difficulties are absent in the Goertler plane, since the continuity equation is absent and the initial profile, regardless of the flow geometry, is the similar flat-plate profile.

The Differential Problem

Of the several problems considered in evaluating the procedure, primary attention was given to the external flow given by $U_e(X^*) = U_o - aX^*$, where U_o and a are constants. This flow was selected because it is of the general, nonsimilar type and yields an adverse pressure gradient with the resulting complication of boundary-layer separation. As flows of this type have been studied previously (*11, 13, 17, 18*), comparison with prior work is possible. The flow can be given the physical interpretation of flow over a negatively inclined plane or of flow over a flat plate in a diverging channel.

An extensive study was made of problems involving coupling between the energy and momentum equations, including the complications of viscous dissipation and variable physical properties, since problems this difficult are of most interest. The difference equations for this class of problems, however, are quite lengthy, even in abbreviated notation. Therefore, to avoid obscuring the basic simplicity of the method, the more general equations are relegated to the Appendix, and this presentation will focus on the simpler case involving constant physical properties. The basic procedure to be illustrated requires relatively little modification for application to the boundary-layer equations in a much more general form.

The two-dimensional form of Prandtl's boundary-layer equations for incompressible flow is

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \tag{1a}$$

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = U_e\frac{dU_e}{dX} + \frac{\partial^2 U}{\partial Y^2} \tag{1b}$$

The Goertler-transformed equation for this case is

$$h_{\beta\beta\beta} + hh_{\beta\beta} + \lambda(\alpha)(1 - h_\beta^2) = 2\alpha(h_\beta h_{\beta\alpha} - h_{\beta\beta}h_\alpha) \tag{2}$$

with the conditions

$$\beta = 0: \quad h = h_\beta = 0 \qquad \beta \to \infty: \quad h_\beta \to 1 \tag{3}$$

The wall boundary conditions correspond to the usual no-slip and no-suction or blowing conditions. At the leading edge of the body Equation (2) reduces to the ordinary differential equation:

$$h_{\beta\beta\beta} + hh_{\beta\beta} = 0 \tag{4}$$

For certain flow geometrics, the stream functions h become independent of α and the principal function λ is constant. For these flow geometries, the Goertler-transformed equation reduces to the Falkner-Skan equation

$$h_{\beta\beta\beta} + hh_{\beta\beta} + \lambda\,(1 - h_\beta^2) = 0 \tag{5}$$

which is also a similar type of equation. For the similar equation and the Falkner-Skan equation, the boundary conditions are given by Equation (3). These two equations allow one to start the numerical solution at the leading *edge* of the nonsimilar region of the flow geometry, thereby eliminating the calculations for similar initial regions.

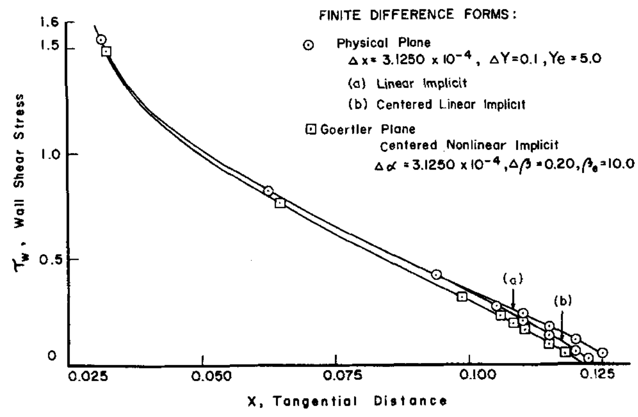

Fig. 2. Wall-shear variation with tangential distance, incompressible, steady state equations.

The compressible-flow problem studied corresponds to the same physical situation but at high velocity, where viscous dissipation and variations in physical properties become important. Two boundary conditions were considered: the simplest case, in which the wall temperature was specified, and the case of the adiabatic wall.

THE DIFFERENCE METHOD

In prior work on difference solutions of nonlinear differential equations, it has often been the practice to linearize the equations at each step in the procedure to facilitate solution of the system of equations. The disadvantage of such a technique lies in the fact that the resulting difference equations are not "centered", and hence the results are less accurate for a given step size than the corresponding nonlinear centered scheme. A number of linear and nonlinear centered methods were studied in this work, and a nonlinear method was devised that holds important advantages over the other methods. The additional computation required to solve the nonlinear algebraic problems was much more than offset by the improvement in accuracy. For a given accuracy the nonlinear method can use larger step sizes so that the amount of computation actually required is less than for linear methods.

The Difference Equations

Consider a difference mesh such that $(l-1)\Delta\alpha = \alpha$ and $(k-1)\Delta\beta = \beta$, where it will be recalled that α denotes distance measured parallel to the boundary and β distance normal to the boundary in the Goertler plane. $l = 1$ denotes the initial position or leading edge and $k = 1$ denotes the boundary.

In the difference procedure the values of the dependent variables are regarded as known along the mesh columns $l, l-1, l-2 \ldots 1$, and the difference equations are used to compute the values of the variables on the mesh column $l+1$. In this way the solution is propagated from the initial profile in the direction of increasing α.

The presentation of the difference equations will be simplified by using parentheses to denote the central divided difference analog of normal direction derivatives; for example (h) denotes $h_{k,l}$ and (h_β) denotes $(h_{l,k+1} - h_{l,k-1})/2\Delta\beta$. The nonlinear difference equations are solved by an iterative matrix-inversion procedure. Primes denote the iteration level and the mesh column, or l level, as indicated below.

1. No prime denotes the known values at the $(l)^{th}$ level.

2. One prime denotes the known values at the $(l+1)^{th}$ level. These correspond to the values of the dependent variables calculated in the previous iteration.

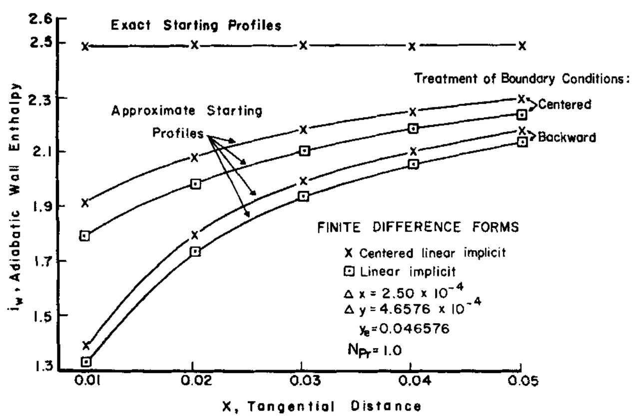

Fig. 3. Adiabatic-wall—enthalpy variation with distance, Howarth-Dorodnitsyn steady state equations.

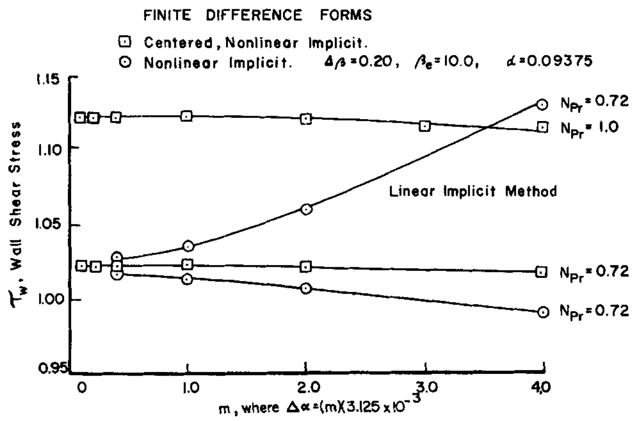

Fig. 4. Wall-shear variation with tangential step size, Goertler incompressible, steady state equations.

3. A double prime denotes the values at the $(l+1)^{th}$ level and the new iteration, that is the unknown values.

The Goertler stream function h depends on α and β. However, at $\alpha = 0$ the equation reduces to the ordinary differential equation, Equation (4), which is solved to yield the initial profile. The difference equation for the initial profile is

$$(h''_{\beta\beta\beta}) + (h')(h''_{\beta\beta}) = 0 \quad (6)$$

The iterative procedure indicated by Equation (6) was found to converge rapidly, independent of the choice of starting values. A linear starting profile was used in most of the work.

The initial profile is propagated downstream with the complete difference equation that approximates the Goertler boundary-layer equations

$$(h''_{\beta\beta\beta}) + (h')(h''_{\beta\beta}) - \lambda'(h'_\beta)(h''_\beta) - \frac{2}{\Delta\alpha}[(\alpha')(h'_\beta) + (\alpha)(h_\beta)](h''_\beta) + \frac{2}{\Delta\alpha}[(\alpha')(h'_{\beta\beta}) + (\alpha)(h_{\beta\beta})](h'') =$$
$$-[(\lambda') + (\lambda)] + (\lambda)(h_\beta)^2 - h_{\beta\beta\beta} - (h)(h_{\beta\beta}) - \frac{2}{\Delta\alpha}[(\alpha')(h'_\beta) + (\alpha)(h_\beta)](h_\beta) + \frac{2}{\Delta\alpha}[(\alpha)(\hat{h}_{\beta\beta}) + (\alpha')(\hat{h}'_{\beta\beta})](h) \quad (7)$$

The Crout-Banachiewicz algorithm (*10*) for a pentadiagonal matrix is used to solve the system of algebraic equations common to the similar equation and the Goertler-transformed equations. Successive matrix inversions, as indicated, lead to rapid convergence of the iterative procedure in all cases. At each step the iteration was terminated when the value of h near the wall changed less than 10^{-6} per iteration. The results of studies using different criteria indicate that the requirement of agreement to within 10^{-6} is unnecessarily restrictive for most purposes. A criterion of 10^{-4} yielded results accurate to at least three significant figures.

Complete solutions including separation required less than 1 min with the Rice University computer. Compressible flow problems required approximately 5 min. There was some difficulty in comparing the data with prior work on different machines; however, it seems clear that the proposed method is at least one order of magnitude faster than previously proposed methods. Speed of computation

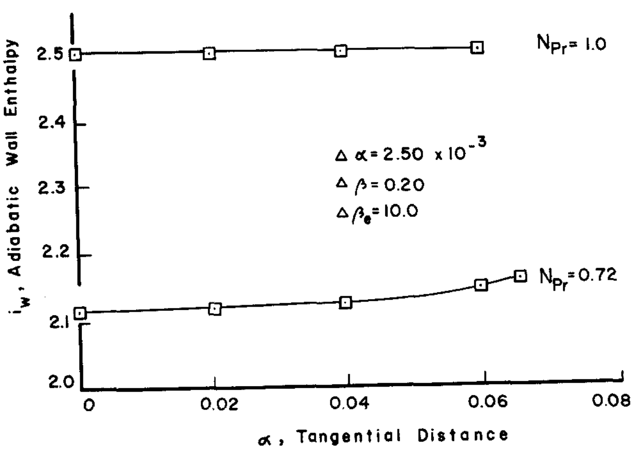

Fig. 5. Adiabatic-wall–enthalpy variation with tangential distance, Goertler compressible, steady state equations.

is, of course, an extremely important factor in problems involving transients or three-dimensional boundary layers. The Rice University computer is similar to the IBM 7040 in speed of computation. The computer time requirements given above are for a $100(\beta)$ by 200 (α) grid.

Treatment of Boundary Conditions

The momentum and energy balances for the untransformed and for the Howarth-Dorodnitsyn–transformed compressible boundary-layer equations are both second-order, nonlinear, partial-differential equations. Linear finite-difference equations used in prior work to approximate these equations can be solved simultaneously at each step with the Crout-Banachiewicz algorithm (*15*) for a tridiagonal matrix. Blottner (*5*) used this technique to solve the equations. For wall boundary conditions of a flux type such as an adiabatic wall condition $i_\beta = 0$, Blottner used a backward difference to approximate the derivatives i_β and $i_{\beta\beta}$. The backward difference analog yielded an expression for the wall enthalpy in terms of parameters evaluated at the first mesh point above the wall. In the present work a difference approximation was developed that satisfies the momentum and energy equation at the boundary and has the additional advantage that the difference approximation for the normal derivative is a central difference, consistent with the difference approximations used throughout the flow field. The method as developed for the untransformed and the Howarth-Dorodnitsyn–transformed compressible equations is given in reference *7* and the application of the method to the Goertler-transformed equations is given in the Appendix. The treatment of the boundary conditions has an important influence on accuracy, as will be shown later.

Stability

The well-known Fourier series method of stability analysis was applied to a variety of numerical procedures. The most important results on the recommended new method are outlined below:

1. The momentum equations for both the incompressible flow and the compressible flow were found to be stable if the u component of the velocity in the boundary layer is always less than or equal to the exterior velocity, that is no "overshoot" within the boundary layer. In all cases the growth terms are of order $\Delta\alpha$, when it is assumed that the principal function is bounded.

2. For problems in compressible flow involving the energy equation, there is an additional requirement that u remain positive. This restriction is of no consequence since the boundary-layer equations are not applicable at or past the separation point.

3. All other methods analyzed seem to have the same or more restrictive requirements for stability.

RESULTS

The most important result from the work, as outlined above, is a new method of solving boundary-layer problems. A number of studies on related problems were carried on in the course of the investigation; for example, several problems in the time-dependent boundary-layer equation were solved. These studies showed that extension of the steady state methods to the transient case presents no great difficulty. Complete details of the transient and other studies are given in reference *7*. The results presented below are all for the steady state.

Incompressible Flow

Figure 1 shows some typical results from a study on accuracy of four methods of solution:

1. The linear, implicit method, in which the coefficients in the difference equation are evaluated at the known α level so that the algebraic procedure is linear at each step. The derivatives with respect to β are approximated by means of functions at the downstream level in a way very much analogous to the Laasonen method (*12*) of solving transient-conduction problems.

2. The nonlinear, implicit method, which is the same as the first method except that the coefficients are evaluated at the downstream α level. Hence, the algebraic procedure is nonlinear at each step.

3. The centered nonlinear implicit method, in which both the coefficients and the approximations to the derivatives are averaged at the two α levels. This method is the one recommended and described in detail earlier.

4. The linear, centered method, which is the same as the recommended method, except that the coefficients are evaluated at the known α level to make the algebraic procedure linear. The averaging of the difference approximations to the derivatives is analogous to the Crank-Nicolson procedure (*3*) for solving linear transient heat conduction or diffusion problems.

The shear stress is shown as a function of α step size in Figure 1, since it is an important function, sensitive to changes in the velocity distribution. Using a large α step has very little effect on the accuracy of the nonlinear centered method, whereas the other methods deviate appreciably even at relatively small values of $\Delta\alpha$. At the largest value of $\Delta\alpha$ shown, the deviation in the centered method is only 0.3%. The related linear method has a

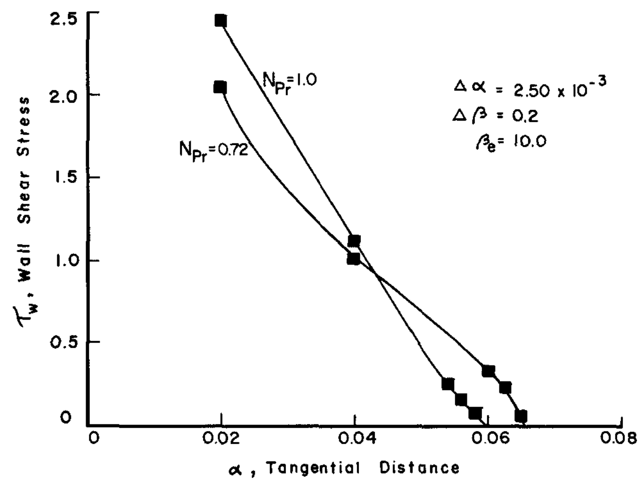

Fig. 6. Wall-shear variation with tangential distance, Goertler compressible, steady state equations.

deviation this large for values of $\Delta\alpha$ 1/10 as large. The linear implicit method has the same deviation with $\Delta\alpha$ 1/20 as large.

The complete shear stress vs. α curve for the incompressible problem is given in Figure 2. In addition to the recommended method, solutions are shown for two related methods in the physical plane. The lowest curve, based on the recommended method, can be regarded as exact, since a very thorough study was made on the error involved. Notice that at $X = 0.121$ the shear stress vanishes corresponding to the point of boundary-layer separation. It was possible to solve the equation up to separation, although the separation point itself is a singular point and the boundary-layer equations become invalid.

Figure 2 shows that the solutions in the physical plane agree with the exact solution except near separation. The form of these solutions, however, is quite sensitive to the initial profile used. As indicated previously, there is no strictly correct initial profile in the physical plane, since the leading edge is a singular point in the equations.

Compressible-Flow Treatment of Boundary Conditions

The Howarth-Dorodnitsyn transformation is used for compressible-flow problems in which it is desirable to stretch the normal distance coordinate. The advantage of the transformation, which is applicable to general flow geometries, lies in the explicit removal of the density from the equations, especially the continuity equation. The system still possesses singularities at the leading edge and at separation. The transformed equations are in the form of the equations for incompressible flow.

Several studies on methods of treating boundary conditions were carried out as outlined in the discussion of the difference method. For conditions corresponding to the adiabatic boundary $i_\beta = 0$, it is somewhat surprising to note that apparently the uncentered backward difference approximation has been used in all prior work involving the boundary-layer equations.

Results of solutions using both the nonlinear implicit method and the centered nonlinear implicit method are shown in Figure 3. Results from three types of solutions are shown: (1) results from the exact solution based on starting profiles from a similar solution; (2) results based on a step function initial enthalpy profile, with boundary conditions treated by the backward method; and (3) results based on the step function initial enthalpy profile with boundary conditions treated by the centered method.

Several conclusions can be formed from inspection of Figure 3. First it can be seen that all the solutions using inexact initial profiles fail to predict the correct adiabatic wall enthalpy. Second, it can be seen that the solutions using the centered method of treating boundary conditions approach the correct wall enthalpy much more quickly than the results based on the backward method. In all cases the deviation from the correct value of 2.5 is due to the singular behavior of the boundary-layer equations at $X = 0$.

Compressible Flow: The Goertler Plane

The Goertler-transformed equations were solved for the compressible-flow problem by several different methods. The wall shear variation with step size is shown in Figure 4 for the four related implicit methods discussed in the incompressible case. Results are included for both the adiabatic and the fixed wall-temperature cases. It can be seen that the nonlinear centered method is much less sensitive to step size than any of the other methods. Also, the number of iterations per step is about one-half the number required in the backward implicit method. Two iteration procedures were used. In the first the momentum and energy equations were advanced simultaneously until constant values were obtained for both. In the second the momentum equation was iterated until steady state was obtained and then, by means of these values, the energy equation was iterated to a steady state. The new energy values were then used to correct the momentum equation. If the momentum value agreed to within 10^{-6} of that calculated before the last energy-equation calculation, the whole procedure was advanced to the next α level; otherwise the procedure was repeated at the same α level. The adiabatic wall condition $i_\beta = 0$ was used, and the enthalpy variation with distance is shown in Figure 5. In this case the correct values are obtained starting at the leading edge as opposed to the result obtained in the Howarth-Dorodnitsyn plane. For a Prandtl number of unity the adiabatic wall enthalpy takes on the constant value 2.5 as predicted precisely by the Busemann relationship (2); however, for $N_{Pr} = 0.72$ the Busemann relationship does not hold, and it is interesting to note that the adiabatic wall enthalpy is not constant but varies slightly with position.

The variation of wall shear with distance for both the specified wall-enthalpy case and the adiabatic-wall case is shown in Figure 6. Comparison of the separation points predicted here, $\alpha_s = 0.0655$ for $N_{Pr} = 0.72$ and $\alpha_s = 0.060$ for $N_{Pr} = 1.0$, shows the influence of the Prandtl number on the separation point. The Mach number also affects the separation point, although it was not studied specifically in this work. The separation point for the compressible-flow problem is upstream from the incompressible-flow separation point.

Typical Profiles

Profiles of h, the stream function variable, at specified positions on the boundary for the incompressible flow are shown in Figure 7. The initial profile denoted by α_5 is shown to indicate clearly the nearly similar behavior of

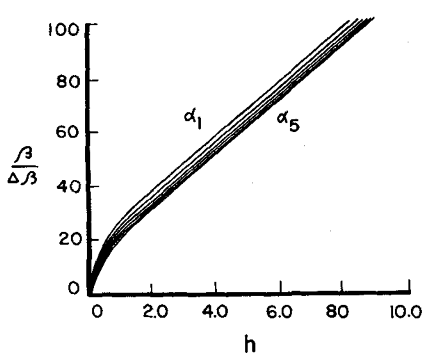

Fig. 7. Stream function profiles, Goertler incompressible, steady state equations. $\Delta\alpha = 6.25 \times 10^{-4}$, $\Delta\beta = 0.1$.

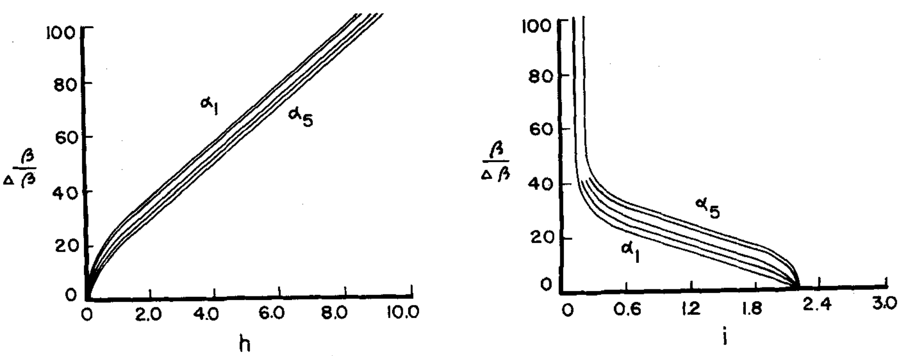

Fig. 8. Stream function and enthalpy profiles, Goertler compressible, steady state equations. $N_{Pr} = 0.72$, $i_\beta|_w = 0.0$, $\Delta\alpha = 6.25 \times 10^{-4}$, $\Delta\beta = 0.1$.

the Goertler variables even up to the separation point, denoted by α_1. This behavior promotes accuracy in the numerical procedure and contrasts with the physical plane, where the profiles change markedly with position.

Figure 8 gives the profiles of h and of i for the corresponding compressible-flow problem with $N_{Pr} = 0.72$ and the adiabatic wall condition. Notice that even in this case the profiles change relatively slowly with position. The marked increase in enthalpy in the boundary layer is the result of viscous heating. It can be seen from the profiles that the solutions were carried out to values of β well beyond the point where the boundary conditions were satisfied. The infinite nature of the boundary conditions caused no difficulty in the numerical procedure.

CONCLUSIONS

Several conclusions from the present work are summarized:

1. The new method for solving the boundary-layer equations applies to the equations in general form. The method is faster and more accurate than methods proposed previously.
2. In the numerical solution of the boundary-layer equations the Goertler plane is preferable to the physical plane or to other transformed planes. In the Goertler plane there is no singularity at the leading edge where the solution of the equation is initiated. The Goertler equations, by also yielding slowly changing profiles of the dependent variable, enhance accuracy.
3. The method of treatment of boundary conditions involving normal derivatives has an important influence on accuracy. A new method is proposed that is more accurate than the methods previously reported.
4. Explicit difference procedures are not recommended for application to boundary-layer problems involving an adverse pressure gradient because of the severe step-size limitation required for stability.
5. Methods of the Hartree-Womersly type are not recommended for general application, although the methods have been used successfully by several workers. Inherent stability and iteration-convergence problems in the methods cast doubt on their applicability to general problems.

ACKNOWLEDGMENTS

This work was supported by the National Science Foundation under Grant GP-661. The support of the Rice University Computer Project is also gratefully acknowledged.

NOTATION

a = constant in the expression for exterior velocity distribution, Equation (1)
c^* = velocity of sound at the reference state
c_p = specific heat
F = viscosity-enthalpy relation, $F = \rho\mu$; Equation (A.7)
F_i = derivative of the viscosity-enthalpy relation, $F_i = dF/di$, Equation (A.8)
$h(\alpha,\beta)$ = stream function of the Goertler transformed equations. For incompressible flow, $h(\alpha, \beta) = (2\alpha)^{-1/2}\, \psi(X, Y)$; for compressible flow, $h(\alpha, \beta) = (2F\ \alpha)^{-1/2}\, \psi(X, Y)$
i = enthalpy per unit mass
k = thermal conductivity
N_{Pr} = Prandtl number
S = constant in Sutherlands viscosity law, Equation (A.7)
T = temperature
U = dimensionless tangential-velocity component parallel to the boundary. For incompressible flow, $U = U^*/U_o$; for compressible flow, $U = U^*/c^*$
$U_e(X)$ = exterior velocity distribution
V = dimensionless normal velocity component $V^* \sqrt{\rho_o/(a\,\mu_o)}$
X = dimensionless tangential distance along the plate. For incompressible flow, $X = X^*\, a/U_o$; for compressible flow, $X = X^* a/c^*$.
Y = dimensionless normal distance from the plate, $Y = Y^* \sqrt{\mu_o/a\,\rho_o}$

Greek Letters

α = tangential distance along the plate, Goertler transformed plane $= \int_o^X U_e(X)dX$
β = normal distance to the plate, Goertler transformed plane. For incompressible flow, $\beta = Y\, U_e(X)(2\alpha)^{-1/2}$; for compressible flow, $\beta = [U_e(X)\, (2F\,\alpha)^{-1/2}] \int_o^Y \rho\, dY$
γ = ratio of specific heats
λ = principal function of the Goertler transformed equations, $\tau(\alpha) = 2\alpha\, d \ln\, (U_e(\alpha))/d\alpha$
ρ = dimensionless density, ρ^*/ρ_o
τ = shear stress. For incompressible flow, $\tau^*\ U_o \sqrt{a\rho_o\mu}$; for compressible flow, $\tau^*[(c^*)^2 a\rho_o\mu_o]^{1/2}$
ψ = stream function in the physical plane. For incompressible flow, $\psi_Y = U$ and $\psi_X = -V$; for compressible flow, $\psi_Y = \rho U$ and $\psi_X = -\rho V$

Subscripts

e = exterior to the boundary layer
k = mesh points in β direction
l = mesh points in α direction
o = constant denoting reference state
s = separation point
w = wall or boundary value
∞ = infinite flow

Superscripts

$*$ = dimensional quantities

LITERATURE CITED

1. Baxter, D. C., and I. Flügge-Lotz, *Tech. Rept. 103*, Div. Eng. Mech., Stanford Univ., Stanford, Calif. (1956).
2. Busemann, A., "Boundary Layer Theory," p. 341, McGraw-Hill, New York (1960).
3. Crank, J., and P. Nicholson, *Proc. Cambridge Phil. Soc.*, **43**, 50 (1947).
4. Dorodnitsyn, A. A., *Prikladnaya Math. Mek.* **6**, (1942).
4. Flügge-Lotz, I., and F. G. Blottner, *Tech. Rept. No. 131*, Div. Eng. Mech., Stanford Univ., Stanford, Calif. (1962).
6. Forsythe, G. E., and W. R. Wasow, "Finite Difference Methods for Partial Differential Equations," Wiley, New York (1960).
7. Fussell, D. D., Ph.D. dissertation, Rice Univ., Houston, Texas (1964).
8. Goertler, H., *J. Math. Mech.*, **6**, 1 (1957).
9. Hartree, D. R., and J. R. Womersley, *Proc. Roy. Soc. (London)*, **A161**, 353 (1937).
10. Hildebrand, F. B., "Introduction to Numerical Analysis," p. 429, McGraw-Hill, New York (1956).
11. Howarth, L., *Proc. Roy. Soc. (London)*, **A194**, 16 (1948).
12. Laasonen, P., "Difference Methods for Initial-Value Problems," p. 93, Interscience, New York (1957).
13. Leigh, D. C. F., *Proc. Cambridge Phil. Soc.*, **51**, 320 (1955).
14. Manohar, R., *Proc. 4th Congr. Theoret. Appl. Mech.*, 135 (1958).
15. Richtmyer, R. D., "Difference Methods for Initial-Value Problems," p. 101, Interscience, New York (1957).
16. Schlichting, Herman, "Boundary Layer Theory," McGraw-Hill, New York (1960).
17. Smith, A. M. O., and D. W. Clutter, *Douglas Aircraft Co. Eng. Paper 1530* (1962).

18. ———, *Douglas Aircraft Co. Eng. Paper 1525* (1963).
19. ———, *Douglas Aircraft Co. Rept. LB 31088* (1963).
20. Wu, J. C., "Proc. 1961 Heat Trans. and Fluid Mech. Inst.," p. 55, Stanford Univ. Press, Stanford, Calif. (1961).

APPENDIX

For compressible flows, the infinite flow conditions that are independent of the coordinate system and the body shape must be known. The Prandtl number Pr, the ratio of the specific heats γ, the constant S of Sutherlands viscosity law (used throughout this work), the temperature T, and the Mach number M are sufficient to describe the infinite flow.

The compressible Goertler-transformed boundary-layer equations as considered in this work combine the Howarth-Dorodnitsyn transformation (4) with the Goertler transformation (8). The Howarth-Dorodnitsyn transformation is incorporated in the transformation, since it stretches the normal direction coordinate, which decreases the magnitude of the normal direction derivatives and the truncation error of the finite difference approximations. The dimensionless compressible Goertler-transformed boundary-layer equations, including viscous dissipation and variable fluid properties, for an ideal gas are

$$h_{\beta\beta\beta} + [(F_i/F)i_\beta + h]h_{\beta\beta} - \lambda(\alpha)\left[\frac{i}{i_e} - (h_\beta)^2\right] = 2\alpha\,[h_\beta\, h_{\beta\alpha} - h_{\beta\beta}\, h_\alpha] \qquad \text{A.1}$$

and

$$i_{\beta\beta}[(F_i/F)i_\beta + h\,Pr]i_\beta + Pr\,U_e^2(\alpha)\left[(h_{\beta\beta})^2 + \lambda(\alpha)\frac{i}{i_e}h_\beta\right] = 2\alpha Pr\,[h_\beta\, i_\alpha - h_\alpha\, i_\beta] \qquad \text{A.2}$$

The exterior velocity distribution, the exterior enthalpy distribution, the principal function, and the wall shear are given by

$$U_e(\alpha) = [U_o - 2\alpha]^{1/2} \qquad \text{A.3}$$

$$i_e(\alpha) = i_o - \tfrac{1}{2}\, U_o + \alpha \qquad \text{A.4}$$

$$\lambda = 2\alpha\frac{d \ln U_e(\alpha)}{d\alpha} \qquad \text{A.5}$$

and

$$\tau_w = \left(\frac{F_w}{2\alpha}\right)^{1/2} U_e^2(\alpha)\, h_{\beta\beta}|_w \qquad \text{A.6}$$

respectively. The viscosity function F and its derivative are given by

$$F = \rho\mu = \frac{\rho_e i_e}{i_o}\,\frac{(1 + S/T_o)}{(S/T_o + i/i_o)}\sqrt{i/i_o} \qquad \text{A.7}$$

and

$$\frac{dF}{di} = \frac{F(S/T_o - i/i_o)}{2i(S/T_o + i/i_o)} \qquad \text{A.8}$$

For similar flows the stream function h is independent of α and the principal function λ is constant. The Goertler-transformed equations reduce to Equations (A.1) and (A.2) with the right-hand sides set equal to zero. The conditions are

$$\beta = 0:\quad h = h_\beta = i_\beta = 0 \qquad \beta \to \infty:\quad h_\beta \to 1,\ i \to i_e(\alpha) \qquad \text{A.9}$$

These equations and conditions are derived and discussed more fully in reference 7.

The Difference Equations

For the flow geometries considered in this work, the exact initial profiles were obtained with similar equations, since the principal function at the leading edge is zero. The recommended nonlinear finite difference equations that approximate the similar equations are

$$(h''_{\beta\beta\beta}) + \left[\frac{F_i'}{F'}(i_\beta') + (h')\right](h''_{\beta\beta}) = 0 \qquad \text{A.10}$$

$$(i''_{\beta\beta}) + \left[\frac{F_i'}{F'}(i_\beta') + Pr(h')\right](i''_\beta) + PrU_e^2(h'_{\beta\beta})^2 = 0 \qquad \text{A.11}$$

The form of the initial profiles that are necessary to start the solution of the difference equations had no effect on the accuracy of the final solution; therefore a linearly increasing stream function profile and a uniform enthalpy profile with magnitude $i_e(o)$ are recommended. When the initial profiles have been obtained, the recommended nonlinear finite-difference equations

$$\begin{aligned}(h''_{\beta\beta\beta}) &+ \left[\frac{F_i'}{F'}(i_\beta') + (h')\right](h''_{\beta\beta}) - (\lambda')(h_\beta')(h_\beta'') \\ &\left[-\frac{(\alpha)}{\Delta\alpha}(h_{\beta\beta}) + \frac{(\alpha')}{\Delta\alpha}(h_\beta')\right](h_\beta'') + \\ &\left[\frac{(\alpha)}{\Delta\alpha}(h_{\beta\beta}) + \frac{(\alpha')}{\Delta\alpha}(h'_{\beta\beta})\right](h'') - (\lambda)(h_\beta)^2 = \\ &-(h_{\beta\beta\beta}) - \left[\left(\frac{F_i}{F}\right)(i_\beta) + (h)\right](h_{\beta\beta}) + \\ &\left[\frac{(\alpha)}{\Delta\alpha}h_\beta + \frac{(\alpha')}{\Delta\alpha}(h_\beta')\right](h_\beta) + \left[\frac{(\alpha)}{\Delta\alpha}(h_{\beta\beta}) + \right. \\ &\left.\frac{(\alpha')}{\Delta\alpha}(h'_{\beta\beta})\right](h) - \frac{(\lambda)(i)}{i_e} - \frac{(\lambda)(i)}{i_e'}\end{aligned} \qquad \text{A.12}$$

$$\begin{aligned}(i''_{\beta\beta}) &+ \left[\left(\frac{F_i'}{F'}\right)(i_\beta') + Pr\left\{(h') + \right.\right. \\ &\left.\left.\frac{(\alpha') + (\alpha)}{\Delta\alpha}[(h') - (h)]\right\}\right](i_\beta'') - \\ &2Pr\,[(\alpha')(h_\beta') + (\alpha)(h)]\,(i'') = -(i_{\beta\beta}) + \\ &2Pr[(\alpha')(h_\beta') + (\alpha)(h_\beta)]\,(i)\left[\left(\frac{F_i}{F}\right)(i_\beta) + \right. \\ &\left. Pr\left\{(h) + \frac{(\alpha') + (\alpha)}{\Delta\alpha}[(h' - h)]\right\}\right](i_\beta) - \\ &Pr\left\{U_e^2(\alpha')(h'_{\beta\beta})^2 + U_e^2(\alpha)(h_{\beta\beta})^2 - \right. \\ &\left.\frac{(i')}{(i_e')}(\lambda')(h_\beta') - \frac{(i)}{(i_e)}(\lambda)(h_\beta)\right\}\end{aligned} \qquad \text{A.13}$$

that approximate the Goertler compressible equations are used to propagate the solution downstream. For the momentum equation one obtains at each mesh column a system of algebraic equations that are solved by the Crout-Banachiewicz algorithm for a pentadiagonal matrix. The energy equation yields at each mesh column a system of algebraic equations that are solved by the Crout-Banachiewicz algorithm for a tridiagonal matrix.

For flux-type boundary conditions such as the adiabatic wall condition, the differential equations must be satisfied at the wall and the difference analogs of all normal direction derivatives preferably should be consistent with the analogs used throughout the flow field, that is central divided differences. Energy equation (A.2), valid at the wall,

$$i_{\beta\beta} + N_{Pr}\, U_e^2(\alpha)\, h^2_{\beta\beta} = 0 \qquad \text{(A.14)}$$

is obtained by utilizing the boundary conditions (A.9) with adiabatic wall condition. The finite difference analog of this equation is obtained by replacing all derivatives by central divided differences. The boundary condition $i_\beta = 0$ is used in the Taylor expansion for $i_{\beta\beta}$ to eliminate the need for image or pseudo points below the boundary.

Manuscript received January 12, 1965; revision received May 14, 1965; paper accepted May 17, 1965. Paper presented at A.I.Ch.E. San Francisco meeting.